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On Empirical Analysis of Gompertzian Mortality Dynamics

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Abstract: The Gompertz mortality is a frequently used two parameter survival distribution model. Standard parameter estimation such as algebraic technique requires deep understanding of its analytical properties together with mortality data for the parameter estimation to be meaningful. Many life offices in Scandinavian economies still use the model in life insurance valuation. Though life table is solely a product of actuarial mortality, its application is not limited to the computation of insurance premiums. The objective of this paper is to estimate the age-dependent mortality rate parameters of Gompertz model. Gompertz assumes that the population of the insured being considered is relatively stable. Because of the fixed change in the instantaneous mortality intensities and Gompertz cases, the life expectancy proportionally influences the associated mortality tables and hence many life insurance products including life annuities are being affected. The observed fixed changes in stable mortality table helps in determining the corresponding changes which may occur with respect to stress testing of life insurance schemes. In this paper, a hypothetical life table has been constructed which can be used to evaluate the life insurance products. Furthermore, some theorems were stated including the superimposition principles directly related to Gompertz and proved as part of our contributions. From the result of the data used, male and female were subjected to mortality rate at 109 to confirm lifespan. While male terminates at 109, female survives the same age till age 112 and thus have longer life span than male counterpart.

Keywords: Gompertz, mortality, non-increasing, probability, parameters, superimposition

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1. Introduction

Following work in [1], a new advanced numerical technique was applied to estimate the parameters in Gompertz model while differentiating males and females so as to construct mortality table. Furthermore, the work in [2] estimated the force of mortality using the numerical technique of first order differential equation. It was observed in [3] that Gompertz model is the most successful law of dying in human mortality actuarially defined as $l_x = kg^{C^x}$ where k, g and C are parameters. In [3]-[7], l_x projects the number of insured surviving to age x. The Authors derived new formulations of Gompertz and applied it to heterogeneous populations leading to stop loss transforms. However, [8] formulated a five-parameter type of Gompertz-Makeham model as a distribution function to allow flexibility. In [9]-[10], the Gompertz model was constructed in terms of modal age at death. In a study carried out in [10]-[12], empirical models for estimating Gompertz mortality with an application to evolution of the Gompertzian slope was investigated. It was observed that the constants are usually estimated from suitable mortality data set using actuarial procedures to allow us construct mortality table. Mortality tables are of two types in actuarial literature: the cohort and the current life tables. In [13]-[17], computing the probability of surviving to a particular age or the remaining life expectancy of the insured at distinct ages constitute some inferences which are drawn from the life table. A cohort defines a collection of lives sharing unifying traits. Cohort life table describes the real mortality experience of a defined collection of lives from birth to the end of life where age specific probabilities are computed using number of death and population size in the current year and since cohort data is refined to a particular point in time, it is actually free of errors. Mortality statistics content of about 100 years in Cohort analysis is necessary which is obtainable only in few populations and which may be unreliable to a high degree. The unavailability and unreliability of the data pose a big threat when constructing a cohort life table for insurance use. Some lives in a collection may have emigrated or died without taking records, making the life expectancy of the collection of lives dead irrelevant. However, the current life table elicits a cross-sectional picture of the mortality and survival pattern at all ages in a population within small time interval but depends wholly on the current age-specific death rates in the year over which it is constructed. The current life table is a standard and valuable technique for comparing mortality data across boundaries and for appraising mortality patterns at the national level. Current life table forecast life span of every member of a hypothetical cohort based on the correct death rates in a defined population. Therefore, the life expectancy of an infant born in a current year, describes the expectation of life which would be determined assuming he is being subjected all through his lifespan to the same age-specific ruling mortality in that current year. The current life table represents imaginary and contrived pattern describing the mortality experience of a true population during any calendar year but, it is an efficient technique of recapitulating facts of mortality and survival patterns of a population so as to develop a sound hypothesis on which actuarial inference about the population are drawn.

Following [12],[14], [18]-[21], it was reported that the probability of dying increases with age and believing that it is true for both modern and historic data. In order to resolve the problem of why mortality seems to follow the logistic model, the [12],[14] studied four models; the logistic model, Gompertz model, Weibull model, and the law of mortality, for effectiveness of modeling mortality at high ages. It was found out in [12],[14] that the logistic model leads to better results on data from 1980 to 1982, from England and Wales thereby outperforming the other three. He further observed that the logistic model fits better for the historical data from these areas and this motivated him to hypothesize several theories which model the probability density function at the highest age. One view was that there is a fixed upper limit to the length of human life. Thatcher analyzed mortality data from different communities of the United Kingdom. Because he was studying historic data profile, he preferred demographic area where he could obtain long historical mortality records. In order to estimate the parameters of the models and fit them to the data, he applied the maximum likelihood estimation technique to analyze the mortality data. Furthermore, [18] studied the set conditions for modeling ageing, which they felt would create more consistency in the research of aging in different fields by conducting five experiments whose results in the field of bio-demography impacts the study of ageing. The authors observed that in reliability theory, different models seem quite applicable when considering biological aging, but noticed that the models lack biological reality. The authors observed models in four research areas: molecular biology, physics, reliability engineering, and evolutionary biology. It was noticed that the results of the experiment in one field area of study would possibly not model the same data using another field's set condition. They thereafter set a convenient biological process which are built from all four fields and through simulations, confirmed the experimentally observed results.

In order to test the fit of logistic models of the force of mortality, [19]-[20] applied data from the human mortality database for females and males aged 25–109 in 14 different countries. He thereafter suggested a new shifting logistic model that would efficiently forecast age-specific rates of adult mortality. The authors compared model to the Lee–Carter method for modeling and forecasting mortality by age and found out that his model dealt with many areas of weaknesses in the Lee–Carter model, this serves as a basis for age-specific mortality predictions. In [21], many actuarial models such as Gompertz, Perks, Polynomial and Wittstein models were applied to examine human mortality pattern.

It was noticed in [22]-[23] that a well-defined actuarial technique for determining quantitatively mortality pattern in a cohort and spotting differences in age-specific mortality within the cohorts is necessary. In [18], explanation was obtained on how Gompertz parameters were actuarially estimated employing linear regression, that possess higher error than the maximum likelihood method, but he appreciated that the application of the maximum likelihood methods gives room for efficient and precise results of the mortality models. Moreover, [24] extended the senescent mortality of Gompertz to 100 years while conducting the adequacy of Taylor's law on Gompertz's, Makeham's and Siler's models.

2 Mathematical Review of Actuarial functions

X is the random lifetime of a new-born.

The distribution of a life aged x denoted as (x) is defined by its survival function: $S_0(x)=\Pr[X>x]$, $S_0(0)=1$, $S_0(\infty)=\lim_{x\to\infty}S_0(x)=0$. Non-increasing function of x does not have high probability of surviving longer duration.

In [25], we observe the cumulative distribution function $F_0(x)=\Pr[X\le x]$ (1)

Non-decreasing $F_0(0)=0$; and $F_0(\infty)=1$ (2)

$$F_0(0)=1-S_0(x) \tag{3}$$

Probability density function: $f_0(x) = \frac{dF_0(x)}{dx} = -\frac{dS_0(x)}{dx}$ (4)

non-negative: $f_0(x)$ for any $x \geq 0$ (5)

$F_0(x) = \int_0^x f_0(z)dz, and, S_0(x) = \int_x^\infty f_0(z)dz$ (6)

The force of mortality for a new born at age x:

$$\mu_x = \frac{f_0(x)}{1 - F_0(x)} = \frac{f_0(x)}{S_0(x)} = -\frac{1}{S_0(x)}\frac{dS_0(x)}{dx} = -\frac{d\log_e S_0(x)}{dx} \quad (7)$$

If μ_x is the mortality intensity of (x) in the calendar year s, then

$$\mu_x(s) = \lim_{h \to 0}\left[\frac{\Pr\left(x < T_0(s-x) \leq x + h \middle| T_0(s-x) > x\right)}{h}\right]$$

In [25] $\mu_x \delta x \approx \Pr[x < X \leq x + \delta x \mid X > x]$ (8)

For small δx, $\mu_x \delta x$ is the probability that a new born who has attained age x dies between x and $x+1$

$$S_0(x) = \exp\left(-\int_0^x \mu_z dz\right) \quad (9)$$

$$f_0(x) = \mu_x \exp\left(-\int_0^x \mu_z dz\right) \quad (10)$$

$$\overset{0}{e}_0 = E[X] = \int_0^\infty x f_0(x)dx = \int_0^\infty S_0(x)dx \quad (11)$$

The RHS of the equation (11) is obtained using integration by parts.

Variance: $Var[X] = E[X^2] - (E[X])^2 = E[X^2] - \left(\overset{0}{e}_0\right)^2$ (12)

For a person now aged x, its future lifetime is $T_x = X - x$. (12a)

For a newborn, $x = 0$, so that we have $T_0 = X$.

p_x refers to the probability that (x) survives for another year.

$q_x = 1 - p_x$, on the other hand, refers to the probability that (x) dies within one year.

3 Material and Methods

3.1 The Gompertz Model

The Gompertz survival model defines a population's mortality rate μ_x with a two-parameter equation defined below. As both childhood and young ages pass, the mortality rates can be described in terms of exponential function. Gompertz first observed that the law of geometric progression pervades after a certain age in many populations and hence modelled the mortality risk exponentially as follows. The Gompertz exponential B shows the level of senescent mortality defining the exponential pattern of mortality at adulthood while C describes the geometric rise in mortality at higher ages

$$\mu_x = BC^x \Rightarrow l_x = \exp\left[-\frac{B}{\log c}\left(c^x - 1\right)\right], x \geq 0, B > 0, c > 1 \tag{13}$$

$$\int_0^x \mu_t dt = \int_0^x BC^t dt \tag{13a}$$

$$\int_0^x \mu_s ds = \left[\frac{Bc^s}{\log_e C}\right]_0^x \Rightarrow \frac{Bc^x}{\log_e C} - \frac{B}{\log_e C} \tag{14}$$

$$\int_0^x \mu_s ds = -\left(C^x - 1\right)\log_e g, where, \log_e g = \frac{-B}{\log_e C} \tag{15}$$

$$\int_0^x \mu_s ds = -\log_e g^{C^{x-1}} \tag{16}$$

$$l_x = l_0 e^{-\int_0^x \mu_t d_t} = l_0 e^{\log_e g^{c^{x-1}}} = l_0 g^{C^x - 1} \tag{17}$$

$$l_x = \frac{l_0}{g} g^{C^x} = kg^{C^x}, k = \frac{l_0}{g} \tag{18}$$

$$_s p_x = \frac{kg^{C^{x+1}}}{kg^{C^x}} =_s p_x = g^{C^x\left(C^t - 1\right)} \tag{19}$$

3.2 Theorem 1

If a mortality table follows Makeham's model, then

(i) $\mu_x = \dfrac{1 - \delta\bar{a}_x - \rho\bar{a}_x}{\bar{a}^{\bullet}_x}$, where $\bar{a}^{\bullet}_x$ is evaluated at $\delta^{\bullet}$ and (ii) $\delta^{\bullet} = \theta s$

Proof

Suppose $\bar{A}_x$ is the continuous whole life assurance, $\bar{a}_x$ is the continuous whole life annuity and δ is the force of interest

$$\bar{A}_x = \int_{-\infty}^{\infty} v^s f_{T(x)}\left(t\right) ds = \int_0^{\Omega - x} e^{-\delta s} {}_s P_x \mu_x\left(s\right) ds \tag{20}$$

$$\mu_x(s) = A + BC^{x+s} \tag{21}$$

$$\overline{A}_x = \int_0^{\Omega-x} e^{-\delta s}\, {}_sP_x\left(\rho + BC^{x+s}\right)ds \tag{22}$$

$$\overline{A}_x = \int_0^{\Omega-x} \rho e^{-\delta s}\, {}_sP_x ds + \int_0^{\Omega-x} e^{-\delta s}\, {}_sP_x BC^{x+s} ds \tag{23}$$

$$\overline{A}_x = \int_0^{\Omega-x} \rho e^{-\delta s}\, {}_sP_x ds + Bc^x \int_0^{\Omega-x} e^{-\delta s}\, {}_sP_x C^s ds \tag{24}$$

$$\overline{A}_x = \int_0^{\Omega-x} \rho e^{-\delta s}\, {}_sP_x ds + BC^x \int_0^{\Omega-x} e^{-\delta s}\, {}_sP_x e^{s\log_e(c)} ds \tag{25}$$

$$\overline{A}_x = \rho \int_0^{\Omega-x} e^{-\delta s}\, {}_sP_x ds + BC^x \int_0^{\Omega-x} {}_sP_x e^{s\left(\log_e(C)-\delta\right)} ds =$$

$$\rho \int_0^{\Omega-x} e^{-\delta s}\, {}_sP_x ds + BC^x \int_0^{\Omega-x} {}_sP_x e^{-\left(s\delta - s\log_e(C)\right)} ds \tag{26}$$

$$\delta^{\bullet} = s\delta - s\log_e C \tag{27}$$

$$\overline{A}_x = \rho \int_0^{\Omega-x} e^{-\delta s}\, {}_sP_x ds + Bc^x \int_0^{\Omega-x} {}_sP_x e^{-\delta^{\bullet}} ds \tag{28}$$

$$\overline{A}_x = \rho \overline{a}_x + \mu_x \overline{a}^{\bullet}_x \tag{29}$$

But $\overline{A}_x = 1 - \delta \overline{a}_x$ (30)

$$1 - \delta\overline{a}_x = \rho\overline{a}_x + \mu_x \overline{a}^{\bullet}_x \Rightarrow \mu_x \overline{a}^{\bullet}_x = 1 - \delta\overline{a}_x - \rho\overline{a}_x \tag{31}$$

$$\mu_x = \frac{1 - \delta\overline{a}_x - \rho\overline{a}_x}{\overline{a}^{\bullet}_x} \tag{32}$$

$$\delta^{\bullet} = s\delta - s\log_e C = \theta s \Rightarrow \theta = \delta - \log_e C \tag{33}$$

If $\delta = 0$ in equation (26),

$$\overline{A}_x = \rho \int_0^{\Omega-x} {}_sP_x ds + BC^x \int_0^{\Omega-x} {}_sP_x e^{-\left(-s\log_e C\right)} ds$$

$$= \rho \mathring{e}_x + BC^x \overline{a}_x\left(\delta^{\bullet\bullet}\right) \tag{33a}$$

$$\overline{A}_x = \rho \mathring{e}_x + \mu_x \overline{a}_x\left(\delta^{\bullet\bullet}\right), \delta^{\bullet\bullet} = -s\log_e C \tag{33b}$$

3.3 Theorem 2

$\left(\overline{\mathrm{A}}_x^{(\delta^\bullet)}\right)_G = \left(\overline{\mathrm{A}}_x^{(\delta)}\right)_M$ where G denotes Gompertz and M , Makeham defined by $\mu_x = A + BC^x$

where A is a constant

Proof

$$\overline{\mathrm{A}}_x = \int_{-\infty}^{\infty} v^t f_{T(x)}(t)\,dt = \int_0^{\Omega - x} e^{-\delta t}\,{}_tP_x\mu_x(s)\,dt \tag{34}$$

$$ {}_tP_x = S^t g^{C^{x+t}-C^x} \tag{35}$$

$$\overline{\mathrm{A}}_x = \int_0^{\Omega - x} e^{-\delta t} S^t g^{C^{x+t}-C^x}\mu_x(t)\,dt = \int_0^{\Omega - x} e^{-\delta t} S^t g^{C^{x+t}-C^x}\mu_x(t)\,dt \tag{36}$$

$$\overline{\mathrm{A}}_x = \int_0^{\Omega - x} e^{t\log_e S - \delta t} g^{C^{x+t}-C^x}\mu_x(t)\,dt = \int_0^{\Omega - x} e^{-t(\delta - \log_e S)} g^{C^{x+t}-C^x}\mu_x(t)\,dt \tag{37}$$

$$\left(\overline{\mathrm{A}}_x\right)_{gompertz} = \int_0^{\Omega - x} e^{-\delta^\bullet t} g^{C^{x+t}-C^x}\mu_x(t)\,dt \tag{38}$$

$$\delta^\bullet = \delta_{\,-\,\log_e S} \tag{39}$$

3.4 Superimposition Principle Theorem

3.5 Theorem 3

This principle tells us that if two different mortality cohorts of same age group have the same force of mortality under Gompetz mortality frame work, then the implication is that the aggregate probability of survival on their combination will not contain the initial mortality parameter.

If $\mu_x = BC^x + AD^x, then,_t\ p_x = h_1^{\left(C^{x+t}-C^x\right)} h_2^{\left(D^{x+t}-D^x\right)}$

Proof

$$\mu_x = BC^x + AD^x \Rightarrow \int_0^x \mu_t\,dt = \int_0^x BC^t + AD^t\,dt = \int_0^x BC^t\,dt + \int_0^x AD^t\,dt \tag{40}$$

$$\int_0^x \mu_t\,dt = \left[\frac{BC^t}{\log_e C}\right]_0^x + \left[\frac{AD^t}{\log_e D}\right]_0^x \tag{41}$$

$$\int_0^x \mu_t\,dt = \frac{BC^x}{\log_e C} - \frac{B}{\log_e C} + \frac{AD^x}{\log_e D} - \frac{A}{\log_e D} \tag{42}$$

$$\int_0^x \mu_t dt = \frac{BC^x}{\log_e C} - \frac{B}{\log_e C} + \frac{AD^x}{\log_e D} - \frac{A}{\log_e D} \quad (43)$$

$$\int_0^x \mu_t dt = \left(\frac{BC^x}{\log_e C} - \frac{B}{\log_e C}\right) + \left(\frac{AD^x}{\log_e D} - \frac{A}{\log_e D}\right) = \frac{B}{\log_e C}\left(C^x - 1\right) + \frac{A}{\log_e D}\left(D^x - 1\right) \quad (44)$$

Let $\log_e h_1 = \frac{-B}{\log_e C}$, $\log_e h_2 = \frac{-A}{\log_e D}$ (45)

$$\int_0^x \mu_t dt = -\log_e h_1 \left(C^x - 1\right) - \log_e h_2 \left(D^x - 1\right) = -\left[\left(C^x - 1\right)\log_e h_1 + \left(D^x - 1\right)\log_e h_2\right] \quad (46)$$

$$\int_0^x \mu_t dt = -\left[\log_e h_1^{\left(C^x - 1\right)} + \log_e h_2^{\left(D^x - 1\right)}\right] \quad (47)$$

$$\int_0^x \mu_t dt = -\left[\log_e h_1^{\left(C^x - 1\right)} h_2^{\left(D^x - 1\right)}\right] \quad (48)$$

$$l_x = l_0 e^{-\int_0^x \mu_t d_t} = l_0 e^{\log_e h_1^{\left(C^x - 1\right)} h_2^{\left(D^x - 1\right)}} = l_0 h_1^{\left(C^x - 1\right)} h_2^{\left(D^x - 1\right)} \quad (49)$$

$$l_x = \frac{l_0}{g} g^{c^x} = kg^{c^x}, k = \frac{l_0}{g} \quad (50)$$

$$_t p_x = \frac{l_0 h_1^{\left(C^{x+t} - 1\right)} h_2^{\left(D^{x+t} - 1\right)}}{l_0 h_1^{\left(C^x - 1\right)} h_2^{\left(D^x - 1\right)}} = \frac{l_0 h_1^{\left(C^{-1}\right)} h_1^{\left(C^{x+t}\right)} h_2^{\left(D^{-1}\right)} h_2^{\left(D^{x+t}\right)}}{h_1^{\left(C^{-1}\right)} l_0 h_1^{\left(C^x\right)} h_2^{\left(D^{-1}\right)} h_2^{\left(D^x\right)}} = \frac{h_1^{\left(C^{x+t}\right)} h_2^{\left(D^{x+t}\right)}}{h_1^{\left(C^x\right)} h_2^{\left(D^x\right)}} = h_1^{\left(C^{x+t} - C^x\right)} h_2^{\left(D^{x+t} - D^x\right)} \quad (51)$$

4 Analysis and Presentation of Results

For the purpose of this study, the data used in this study came mainly from the mortality of the population of England and Wales during the years 1990, 1991 and 1992.

4.1 Results for Males

$$C = 1.086164248 \quad (52)$$

$$g = 0.9995969509 \quad (53)$$

$$k = 91840.21055 \quad (54)$$

$$\mu_x = 0.000006284487297\left(1.086164248\right)^x \quad (55)$$

$$l_x = (91840.210546049)0.9995969509^{(1.086164248)^x}$$

4.2 Results for Females

$$C = 1.097489964$$

$$g = 0.9998729085$$

$$k = 94880.08259 \quad (56)$$

$$\mu_x = 0.000002230056027(1.097489964)^x \quad (57)$$

$$l_x = (94880.08259)0.9998729085^{(1.097489964)^x} \quad (58)$$

The life table computed below is assumed to describe the mortality level from age 20 years till the end of mortality table and for every individual age, the corresponding risk of death is given. When mortality data are available, actuarial computations can be performed conveniently. However, the availability of mortality data varies with ages where older ages are well covered when compared with lower. The mortality data was sourced from the mortality of the population of England and Wales during the years 1990, 1991 and 1992 because the data was believed to have been validated and hence will be more reliable. Furthermore, there is no available mortality data which can be sourced locally. This occurs because there is no vital registration system which can continuously collect reliable information.

4.3 Males Mortality Table

Table 1: Male Mortality Table based on Gompertz Model

x	l_x	d_x	q_x	p_x	L_x	T_x	e_x	$\mathring{e}_x$	$x+e_x$	μ_x
20	91647	17	0.00018	0.9998	91639	6200918	67.67	68.167	87.67	3.28E-05
21	91630	18	0.0002	0.9998	91621	6109279	66.68	67.1796	87.68	3.57E-05
22	91612	20	0.00021	0.9998	91603	6017658	65.69	66.1931	87.69	3.87E-05
23	91593	21	0.00023	0.9998	91582	5926055	64.71	65.2076	87.71	4.21E-05
24	91571	23	0.00025	0.9997	91560	5834473	63.72	64.223	87.72	4.57E-05
25	91548	25	0.00027	0.9997	91536	5742913	62.74	63.2395	87.74	4.96E-05
26	91523	27	0.0003	0.9997	91510	5651377	61.76	62.2572	87.76	5.39E-05
27	91496	30	0.00032	0.9997	91481	5559868	60.78	61.2761	87.78	5.85E-05
28	91466	32	0.00035	0.9996	91450	5468386	59.8	60.2963	87.8	6.36E-05
29	91434	35	0.00038	0.9996	91417	5376936	58.82	59.3178	87.82	6.91E-05
30	91399	38	0.00042	0.9996	91380	5285519	57.84	58.3408	87.84	7.50E-05
31	91361	41	0.00045	0.9996	91341	5194139	56.87	57.3654	87.87	8.15E-05

32	91320	45	0.00049	0.9995	91298	5102798	55.89	56.3917	87.89	8.85E-05
33	91276	48	0.00053	0.9995	91251	5011500	54.92	55.4197	87.92	9.61E-05
34	91227	53	0.00058	0.9994	91201	4920249	53.95	54.4496	87.95	0.000104
35	91175	57	0.00063	0.9994	91146	4829048	52.98	53.4815	87.98	0.000113
36	91117	62	0.00068	0.9993	91086	4737902	52.02	52.5155	88.02	0.000123
37	91055	67	0.00074	0.9993	91022	4646815	51.05	51.5517	88.05	0.000134
38	90988	73	0.0008	0.9992	90952	4555793	50.09	50.5903	88.09	0.000145
39	90915	79	0.00087	0.9991	90875	4464842	49.13	49.6315	88.13	0.000158
40	90836	86	0.00095	0.9991	90793	4373966	48.18	48.6753	88.18	0.000171
41	90750	93	0.00103	0.999	90703	4283174	47.22	47.7219	88.22	0.000186
42	90656	101	0.00112	0.9989	90606	4192471	46.27	46.7716	88.27	0.000202
43	90555	110	0.00121	0.9988	90500	4101865	45.32	45.8244	88.32	0.00022
44	90445	119	0.00132	0.9987	90386	4011365	44.38	44.8806	88.38	0.000239
45	90326	129	0.00143	0.9986	90261	3920979	43.44	43.9403	88.44	0.000259
46	90197	140	0.00156	0.9984	90127	3830718	42.5	43.0037	88.5	0.000281
47	90057	152	0.00169	0.9983	89981	3740591	41.57	42.0711	88.57	0.000306
48	89904	165	0.00183	0.9982	89822	3650610	40.64	41.1427	88.64	0.000332
49	89740	179	0.00199	0.998	89650	3560788	39.72	40.2187	88.72	0.000361
50	89561	194	0.00216	0.9978	89464	3471138	38.8	39.2993	88.8	0.000392
51	89367	210	0.00235	0.9977	89262	3381674	37.88	38.3847	88.88	0.000426
52	89157	227	0.00255	0.9974	89043	3292412	36.98	37.4753	88.98	0.000462
53	88930	246	0.00277	0.9972	88807	3203368	36.07	36.5713	89.07	0.000502
54	88683	267	0.00301	0.997	88550	3114562	35.17	35.673	89.17	0.000545
55	88416	289	0.00327	0.9967	88272	3026012	34.28	34.7806	89.28	0.000592
56	88127	313	0.00355	0.9965	87971	2937740	33.39	33.8944	89.39	0.000643
57	87815	338	0.00386	0.9961	87645	2849769	32.51	33.0148	89.51	0.000699
58	87476	366	0.00419	0.9958	87293	2762124	31.64	32.142	89.64	0.000759
59	87110	396	0.00455	0.9955	86912	2674831	30.78	31.2763	89.78	0.000824

60	86714	428	0.00494	0.9951	86500	2587919	29.92	30.4182	89.92	0.000895
61	86286	463	0.00536	0.9946	86055	2501419	29.07	29.5678	90.07	0.000973
62	85823	500	0.00582	0.9942	85574	2415364	28.23	28.7256	90.23	0.001056
63	85324	539	0.00632	0.9937	85054	2329790	27.39	27.8919	90.39	0.001147
64	84784	582	0.00686	0.9931	84493	2244736	26.57	27.067	90.57	0.001246
65	84202	628	0.00745	0.9925	83889	2160243	25.75	26.2513	90.75	0.001354
66	83575	676	0.00809	0.9919	83237	2076354	24.95	25.4452	90.95	0.00147
67	82898	728	0.00879	0.9912	82534	1993118	24.15	24.649	91.15	0.001597
68	82170	784	0.00954	0.9905	81778	1910584	23.36	23.8631	91.36	0.001734
69	81386	843	0.01036	0.9896	80964	1828806	22.59	23.0878	91.59	0.001884
70	80543	906	0.01125	0.9888	80090	1747841	21.82	22.3234	91.82	0.002046
71	79637	972	0.01221	0.9878	79151	1667751	21.07	21.5705	92.07	0.002223
72	78665	1043	0.01325	0.9867	78144	1588600	20.33	20.8293	92.33	0.002414
73	77622	1117	0.01439	0.9856	77064	1510456	19.6	20.1001	92.6	0.002622
74	76505	1195	0.01562	0.9844	75908	1433393	18.88	19.3833	92.88	0.002848
75	75311	1277	0.01695	0.983	74672	1357485	18.18	18.6793	93.18	0.003093
76	74034	1362	0.0184	0.9816	73353	1282813	17.49	17.9883	93.49	0.00336
77	72672	1451	0.01997	0.98	71946	1209460	16.81	17.3106	93.81	0.003649
78	71220	1543	0.02167	0.9783	70449	1137514	16.15	16.6467	94.15	0.003964
79	69677	1639	0.02352	0.9765	68858	1067065	15.5	15.9966	94.5	0.004305
80	68039	1736	0.02552	0.9745	67171	998207	14.86	15.3608	94.86	0.004676
81	66302	1836	0.02768	0.9723	65385	931037	14.24	14.7394	95.24	0.005079
82	64467	1936	0.03003	0.97	63499	865652	13.63	14.1325	95.63	0.005517
83	62531	2037	0.03258	0.9674	61512	802153	13.04	13.5405	96.04	0.005992
84	60494	2138	0.03534	0.9647	59425	740641	12.46	12.9635	96.46	0.006509
85	58356	2236	0.03832	0.9617	57238	681216	11.9	12.4015	96.9	0.007069
86	56120	2332	0.04155	0.9584	54954	623978	11.35	11.8546	97.35	0.007679
87	53788	2423	0.04505	0.9549	52576	569024	10.82	11.3228	97.82	0.00834

88	51365	2509	0.04884	0.9512	50110	516447	10.31	10.8062	98.31	0.009059
89	48856	2586	0.05293	0.9471	47563	466337	9.8	10.3046	98.8	0.009839
90	46270	2654	0.05736	0.9426	44943	418774	9.32	9.81785	99.32	0.010687
91	43616	2711	0.06215	0.9379	42261	373831	8.85	9.34578	99.85	0.011608
92	40906	2754	0.06732	0.9327	39529	331570	8.39	8.88805	100.4	0.012608
93	38152	2781	0.0729	0.9271	36761	292041	7.94	8.44422	100.9	0.013695
94	35371	2792	0.07893	0.9211	33975	255279	7.51	8.01375	101.5	0.014875
95	32579	2783	0.08543	0.9146	31188	221304	7.1	7.59592	102.1	0.016156
96	29796	2754	0.09244	0.9076	28419	190117	6.69	7.18982	102.7	0.017548
97	27042	2704	0.09999	0.9	25690	161698	6.29	6.79427	103.3	0.01906
98	24338	2631	0.10812	0.8919	23022	136008	5.91	6.40775	103.9	0.020703
99	21706	2537	0.11687	0.8831	20438	112986	5.53	6.02829	104.5	0.022487
100	19169	2421	0.12628	0.8737	17959	92548.5	5.15	5.65332	105.2	0.024424
101	16749	2284	0.13639	0.8636	15607	74589.4	4.78	5.27938	105.8	0.026529
102	14464	2130	0.14723	0.8528	13400	58982.9	4.4	4.90184	106.4	0.028815
103	12335	1959	0.15885	0.8412	11355	45583.3	4.01	4.51434	107	0.031297
104	10375	1777	0.1713	0.8287	9487	34228.2	3.61	4.10798	107.6	0.033994
105	8598	1587	0.1846	0.8154	7805	24741.4	3.17	3.67012	108.2	0.036923
106	7011	1394	0.19882	0.8012	6314	16936.8	2.68	3.18243	108.7	0.040105
107	5617	1202	0.21397	0.786	5016	10622.9	2.12	2.61775	109.1	0.04356
108	4415	1016	0.23011	0.7699	3907	5606.75	1.43	1.93499	109.4	0.047313
109	3399	3399	1	0	1700	1699.59	1	1.5	110	0.05139

4.4 Female Mortality Table

Table 2: Female Mortality Table based on Gompertz Model

X	l_x	d_x	q_x	p_x	L_x	T_x	e_x	$\mathring{e}_x$	$x+e_x$	μ_x
20	94803	8	7.96E-05	0.99992	94799	6661665	70.27	70.8	90	1.43E-05

21	94795	8	8.74E-05	0.999913	94791	6566866	69.28	69.8	90	1.57E-05
22	94787	9	9.59E-05	0.999904	94782	6472075	68.28	68.8	90	1.73E-05
23	94778	10	0.0001053	0.999895	94773	6377293	67.29	67.8	90	1.89E-05
24	94768	11	0.0001155	0.999884	94762	6282520	66.3	66.8	90	2.08E-05
25	94757	12	0.0001268	0.999873	94751	6187758	65.31	65.8	90	2.28E-05
26	94745	13	0.0001392	0.999861	94738	6093007	64.31	64.8	90	2.50E-05
27	94732	14	0.0001527	0.999847	94724	5998269	63.32	63.8	90	2.75E-05
28	94717	16	0.0001676	0.999832	94709	5903545	62.33	62.8	90	3.02E-05
29	94701	17	0.0001839	0.999816	94693	5808835	61.34	61.8	90	3.31E-05
30	94684	19	0.0002019	0.999798	94674	5714143	60.36	60.9	90	3.63E-05
31	94665	21	0.0002216	0.999778	94654	5619469	59.37	59.9	90	3.99E-05
32	94644	23	0.0002431	0.999757	94632	5524815	58.38	58.9	90	4.38E-05
33	94621	25	0.0002668	0.999733	94608	5430182	57.4	57.9	90	4.80E-05
34	94595	28	0.0002929	0.999707	94582	5335574	56.41	56.9	90	5.27E-05
35	94568	30	0.0003214	0.999679	94553	5240993	55.43	55.9	90	5.79E-05
36	94537	33	0.0003527	0.999647	94521	5146440	54.45	55	90	6.35E-05
37	94504	37	0.0003871	0.999613	94486	5051919	53.47	54	90	6.97E-05
38	94467	40	0.0004248	0.999575	94447	4957434	52.49	53	90	7.65E-05
39	94427	44	0.0004663	0.999534	94405	4862986	51.51	52	91	8.39E-05
40	94383	48	0.0005117	0.999488	94359	4768581	50.54	51	91	9.21E-05
41	94335	53	0.0005616	0.999438	94308	4674222	49.56	50.1	91	0.000101
42	94282	58	0.0006163	0.999384	94253	4579914	48.59	49.1	91	0.000111
43	94224	64	0.0006764	0.999324	94192	4485661	47.62	48.1	91	0.000122
44	94160	70	0.0007423	0.999258	94125	4391469	46.66	47.2	91	0.000134
45	94090	77	0.0008146	0.999185	94052	4297343	45.69	46.2	91	0.000147
46	94014	84	0.000894	0.999106	93972	4203291	44.73	45.2	91	0.000161
47	93930	92	0.0009811	0.999019	93883	4109320	43.77	44.3	91	0.000177
48	93837	101	0.0010767	0.998923	93787	4015436	42.81	43.3	91	0.000194

49	93736	111	0.0011816	0.998818	93681	3921649	41.86	42.4	91		0.000213
50	93626	121	0.0012967	0.998703	93565	3827968	40.91	41.4	91		0.000234
51	93504	133	0.0014231	0.998577	93438	3734404	39.97	40.5	91		0.000256
52	93371	146	0.0015617	0.998438	93298	3640966	39.03	39.5	91		0.000281
53	93225	160	0.0017138	0.998286	93145	3547668	38.09	38.6	91		0.000309
54	93066	175	0.0018807	0.998119	92978	3454522	37.15	37.7	91		0.000339
55	92891	192	0.0020639	0.997936	92795	3361544	36.23	36.7	91		0.000372
56	92699	210	0.0022649	0.997735	92594	3268750	35.3	35.8	91		0.000408
57	92489	230	0.0024854	0.997515	92374	3176156	34.38	34.9	91		0.000448
58	92259	252	0.0027274	0.997273	92133	3083782	33.47	34	91		0.000492
59	92007	275	0.0029929	0.997007	91870	2991649	32.56	33.1	92		0.000539
60	91732	301	0.0032842	0.996716	91581	2899779	31.66	32.2	92		0.000592
61	91431	329	0.0036037	0.996396	91266	2808198	30.77	31.3	92		0.00065
62	91101	360	0.0039544	0.996046	90921	2716932	29.88	30.4	92		0.000713
63	90741	394	0.0043391	0.995661	90544	2626010	29	29.5	92		0.000783
64	90347	430	0.0047611	0.995239	90132	2535466	28.13	28.6	92		0.000859
65	89917	470	0.005224	0.994776	89682	2445334	27.27	27.8	92		0.000943
66	89447	513	0.0057318	0.994268	89191	2355652	26.41	26.9	92		0.001035
67	88935	559	0.0062889	0.993711	88655	2266461	25.56	26.1	93		0.001135
68	88375	610	0.0068998	0.9931	88071	2177806	24.73	25.2	93		0.001246
69	87766	664	0.00757	0.99243	87433	2089735	23.9	24.4	93		0.001368
70	87101	723	0.0083049	0.991695	86740	2002302	23.08	23.6	93		0.001501
71	86378	787	0.0091108	0.990889	85984	1915562	22.28	22.8	93		0.001647
72	85591	855	0.0099946	0.990005	85163	1829578	21.48	22	93		0.001808
73	84735	929	0.0109636	0.989036	84271	1744415	20.7	21.2	94		0.001984
74	83806	1008	0.012026	0.987974	83303	1660144	19.93	20.4	94		0.002177
75	82799	1092	0.0131906	0.986809	82253	1576841	19.17	19.7	94		0.00239
76	81706	1182	0.0144672	0.985533	81115	1494589	18.43	18.9	94		0.002623

77	80524	1278	0.0158664	0.984134	79886	1413473	17.69	18.2	95	0.002878
78	79247	1379	0.0173997	0.9826	78557	1333588	16.98	17.5	95	0.003159
79	77868	1486	0.0190797	0.98092	77125	1255030	16.27	16.8	95	0.003467
80	76382	1598	0.0209202	0.97908	75583	1177905	15.58	16.1	96	0.003805
81	74784	1715	0.0229361	0.977064	73927	1102322	14.91	15.4	96	0.004176
82	73069	1837	0.0251438	0.974856	72150	1028396	14.25	14.8	96	0.004583
83	71232	1963	0.027561	0.972439	70250	956245	13.61	14.1	97	0.00503
84	69269	2092	0.030207	0.969793	68222	885995	12.99	13.5	97	0.00552
85	67176	2224	0.0331026	0.966897	66064	817773	12.38	12.9	97	0.006058
86	64952	2356	0.0362706	0.963729	63774	751709	11.79	12.3	98	0.006649
87	62597	2487	0.0397354	0.960265	61353	687934	11.21	11.7	98	0.007297
88	60109	2616	0.0435237	0.956476	58801	626581	10.66	11.2	99	0.008009
89	57493	2740	0.0476642	0.952336	56123	567780	10.12	10.6	99	0.00879
90	54753	2857	0.0521876	0.947812	53324	511657	9.6	10.1	100	0.009646
91	51895	2965	0.0571273	0.942873	50413	458333	9.09	9.59	100	0.010587
92	48931	3059	0.062519	0.937481	47401	407920	8.61	9.11	101	0.011619
93	45872	3138	0.0684008	0.931599	44303	360519	8.14	8.64	101	0.012752
94	42734	3197	0.0748136	0.925186	41135	316216	7.69	8.19	102	0.013995
95	39537	3234	0.0818007	0.918199	37920	275081	7.25	7.75	102	0.015359
96	36303	3246	0.0894084	0.910592	34680	237161	6.84	7.34	103	0.016857
97	33057	3229	0.0976851	0.902315	31442	202481	6.44	6.94	103	0.0185
98	29828	3182	0.1066822	0.893318	28237	171039	6.06	6.56	104	0.020303
99	26646	3103	0.1164532	0.883547	25094	142802	5.69	6.19	105	0.022283
100	23543	2991	0.1270538	0.872946	22047	117708	5.34	5.84	105	0.024455
101	20552	2847	0.1385415	0.861459	19128	95661	5	5.5	106	0.026839
102	17704	2673	0.1509752	0.849025	16368	76533	4.68	5.18	107	0.029456
103	15031	2471	0.1644147	0.835585	13796	60165	4.36	4.86	107	0.032328
104	12560	2247	0.1789196	0.82108	11436	46370	4.05	4.55	108	0.035479

105	10313	2006	0.1945489	0.805451	9310	34933	3.75	4.25	109	0.038938
106	8306	1756	0.2113598	0.78864	7429	25624	3.45	3.95	109	0.042734
107	6551	1503	0.2294059	0.770594	5799	18195	3.14	3.64	110	0.0469
108	5048	1256	0.2487365	0.751264	4420	12396	2.8	3.3	111	0.051473
109	3792	1022	0.2693939	0.730606	3282	7975	2.43	2.93	111	0.056491
110	2771	807	0.291412	0.708588	2367	4694	1.98	2.48	112	0.061998
111	1963	618	0.3148137	0.685186	1654	2327	1.41	1.91	112	0.068042
112	1345	1345	1	0	673	673	1	1.5	113	0.074676

5 Discussion of results

The actuarial assumption over which life table is constructed, takes a trend of assumed mortality rate at every integral age, and provided that such a sequence of rates and an arbitrary l_x exist, then l_x column can be computed at integral values of x. Gompertz model has been formulated as an analytical function to compute death probabilities but a reliable estimation technique is necessary to determine the parameters of the model. The function l_x is assumed to be positive and non-increasing and defines the numbers of insured who are expected to survive to age x. From the life tables constructed, at age 20, males have a q_x value of 0.000181, while females have a q_x value of 0.0000796339. The observed trend shows that females have lower probability of mortalities than males as a result of mortality strain experienced by both sexes. From the tables, it seems that there is a sudden change in the number of deaths $-\Delta l_x$ at consecutive ages as from age 50 for both male and female. However, at ages 109, the males have $q_x = 1$, while for females $q_x = 0.269393907$. This age has critical implication as it is the smallest age beyond which no male exists thus all males are assumed terminated. However, females are still expected to continue surviving beyond age 109 for three more years. Based on both life tables constructed, males were observed to have higher q_x values than females. Whereas males lived up to 109 years, females could live up to 112 years. In order to obtain an actuarial technique to life insurance contract meant to ease out the financial strain and its consequences at death, an empirical mortality table based on Gompertz' law of mortality has been constructed.

6 Conclusion

A Properly constructed actuarial model could be a useful instrument in coping with a wide range of mathematical analysis associated with mortality tables. This is because the insured population mortality estimation is important to life office's calculation of expected liability in satisfying the regulatory requirements so as to compete for market shares. In this paper, we present Gompertz mortality model based on England and Wales insured population data with a focus on gaining insight into mortality analysis by addressing analytical techniques of estimating its parameters. The use of the hypothetical mortality table in pricing life assurances may offer a better view of a gradual directional change in mortality leading to a complex form of evolution. These are the expected death rates that have been projected to estimate life and pension obligations. Insurance regulatory framework usually specifies guidelines on mortality rates and assumptions to follow because assumptions are critical in terms of pricing. The estimated life expectancy is used to compute the long-term obligations of life fund. Low mortality assumptions suggest that long term liability of pension funds could be overestimated, however, high assumptions would indicate that life expectancy of the pension scheme will be underestimated and hence underestimating the obligations of pension fund and life insurance providers. From the observations made from both life tables constructed, we can see that the male population was observed to suffer higher mortality rates than the female population, this was evidenced by the values of q_x for males which were significantly higher than the values of q_x obtained for females. The p_x values which indicated the probability that a person's exact age x will live within one year for males were lower than those of females which further proves that the male population suffers higher rate of mortality than females. Furthermore, the T_x signifies the number of person-years lived after exact age x, the values obtained for males were lower than the values obtained for females indicating that males had shorter person-years to live when compared to females. A closer look at the sex differential in mortality by distribution of age shows that females have lower mortality at all ages and females were observed to live longer up to 112 years while males short live only to 109 years. When Gompertz law is applied, we discovered that C-value for males is lower than that of females while the B-value for male is higher than that of female. It is important to note that the life tables meant to be used for life insurance valuation may not have same B and C values as our hypothetical tables because margins usually added to the basic experience year of valuation table would reduce the proportion of mortality rates when age advances.

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