EXACT MODELS FOR ANISOTROPIC FLUID SPHERE

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ABSTRACT: Two categories of exact solutions are found to the Einstein field equations for an anisotropic fluid sphere with a particular choice of the anisotropic factor and one of the gravitational potentials. The condition of pressure isotropy is reduced to alinearsecond order differential equationwhich can be solved in general.Consequentlywe can find exact solutions to the Einstein field equations correspondingto a static spherically symmetric gravitational potential in terms of elementaryfunctions, namely polynomials and product of polynomials andalgebraic functions.These solutions contain particular solutions found previously includingmodels of isotropic relativistic spheres

Keywords: Exact solutions,Einstein field equations,Relativistic spheres

1. INTRODUCTION

Exact solutions to the Einstein field equations with anisotropic matter have been studied by many investigators in recent years. Such Solutions for static spherically symmetric interior spacetimes are important in describing compact objects in relativistic astrophysics. Researchers have attempted to introduce different approaches of finding solutions to the field equations. Hansraj and Maharaj (2006) found solutions to the Einstein-Maxwell system with a specified form of the electrical field with isotropic pressures. These solutions satisfy a barotropic equation of state and regain the Finch and Skea (1989) model. Some of the researchers considered anisotropic pressures in the presence of the electromagnetic field with the linear equation of state of strange stars with quark matter. The approach of Esculpi and Aloma (2010) is interesting in that it utilizes the existence of a conformal symmetry in the spacetime manifold to find a solution. These exact solutions are relevant in the description ofdense relativistic astrophysical objects.

In order to integrate the field equations, various restrictions have been placed by investigators on the geometry of space time and the matter content. Mainly two distinct procedures have been adopted to solve these equations for spherically symmetric static models. Firstly, the coupled differential equations are solved by computation after choosing an equation of state. Secondly, the exact Einstein solutions can be obtained by specifying the geometry and the form of the anisotropic factor. The later technique has been used by Takisa and Maharaj (2013) to produce solutions in terms of special functions and elementary functions that are suitable for the description of relativistic charged stars.

The principal objective of this work istwofold. Firstly, we seek to model a relativistic sphere with anisotropic matter which is physically acceptable. We require that the gravitational fields and matter variables are finite, continuous and well behaved in the stellar interior and the solution is stable with respect to radial perturbations. Secondly, we seek to regain an isotropic solution of Einstein field equations which satisfy the relevant physical criteria when the anisotropy factor vanishes. This ideal is not easy to achieve in practice and only a few examples with the required two features have been found thus far. The main objective of this paper is to provide systematically a solution to Einstein equations with anisotropic matter whish satisfy the above two conditions. In Section 2, the Einstein field equations for the static spherically symmetric line element with anisotropic matter is expressed as an equivalent set of differential equations utilizing a transformation. We chose particular forms for one of the gravitational potentials and the anisotropic factor, which enables us to obtain the condition of pressure anisotropy in the remaining gravitational potential. This is the master equation which determines the solvability of the entire system. Exact solutions to the Einstein field equations in terms of a series and elementary functions are provided in section 3. Physical properties of the solutions are briefly discussed in Section 4.

2. METHODOLOGY

Assume that the interior of a relativistic star should be spherically symmetric. Therefore there exists coordinates$t$time and (*r, θ, φ*)spherical coordinates such that the line element is of the form

$$ds^2 = -e^{2\mu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + sin^2\theta d\varphi^2) \quad (1)$$

where$\mu(r)$ and $\lambda(r)$ are arbitrary functions related to the gravitational potentials. The system of Einstein field equations becomes for the line element (1)

$$\rho = \frac{1}{r^2}\left(1 - e^{-2\lambda}\right) + \frac{2e^{-2\lambda}}{r}\frac{d\lambda}{dr} \quad (2a)$$

$$p_r = -\frac{1}{r^2}\left(1 - e^{-2\lambda}\right) + \frac{2e^{-2\lambda}}{r}\frac{d\mu}{dr} \quad (2b)$$

$$p_t = e^{-2\lambda}\left[\frac{d^2\mu}{dr^2} + \left(\frac{d\mu}{dr}\right)^2 + \frac{1}{r}\left(\frac{d\mu}{dr}\right) - \frac{d\mu}{dr}\frac{d\lambda}{dr} - \frac{1}{r}\left(\frac{d\lambda}{dr}\right)\right] \quad (2c)$$

$$\nabla = p_t - p_r \tag{2d}$$

The quantity ρ is the energy density, p_r is radial the pressure,$p_t$is the tangential pressure and $\nabla$is the anisotropic factor. This system governs the behaviour of the gravitational field for an anisotropic perfect fluid. To solve thissystem it is necessary to choose two of the variables.

At this point we could choose a barotropic equation of state $p_r = p_r(\rho)$.Howeverthis is an approach that we intend to follow in future work.In this approach $\lambda$and $\nabla$are specified. The remaining unknowns are then obtained from the rest of the system.

It is convenient at this point to introduce the choices

$$e^{2\lambda(r)} = \frac{1 - Kr^2}{1 - r^2}, \quad \nabla = \frac{\alpha K r^2}{(1 - Kr^2)^2}$$

where*K* and α are arbitrary constants. We have chosen the above form for the anisotropic factor as it providesfor a wider range of possibilities than the other solutions, and it does produces isotropic and anisotropic solutions which are necessary for a realisticmodel.

On using these choices, the equations (*2b*)-(*2d*)becomes

$$(1 - r^2)\left[\frac{d^2\mu}{dr^2} + \left(\frac{d\mu}{dr}\right)^2 - \frac{1}{r}\frac{d\mu}{dr}\right] - \frac{r(1-K)}{(1-Kr^2)}\left(\frac{1}{r} + \frac{d\mu}{dr}\right) + (1-K) - \frac{\alpha K r^2}{(1-Kr^2)^2} = 0 \tag{3}$$

which is a nonlinear differential equation

To linearisethis equation it is now convenient to introduce the transformation

$$\Psi(x) = e^{2\mu(r)}, x^2 = 1 - r^2$$

With this transformation, the nonlinear differential equation(3) becomes

$$(1 - K + Kx^2)\frac{d^2\Psi}{dx^2} - Kx\frac{d\Psi}{dx} + K(K - \alpha - 1)\Psi = 0 \tag{4}$$

Note that the Einstein system (2) implies

$$\rho = \frac{(1-K)(3-K+Kx^2)}{(1-K+Kx^2)^2}$$

$$p_r = \frac{1}{(1-K+Kx^2)}\left(-\frac{2\mathrm{L}x}{\Psi}\frac{d\Psi}{dx} + K - 1\right)$$

$$p_t = \frac{1}{(1-K+Kx^2)}\left(-\frac{2\mathrm{L}x}{\Psi}\frac{d\Psi}{dx} + K - 1\right) + \frac{\alpha K(1-x^2)}{(1-K+Kx^2)^2}$$

in terms of the independent variable *x*and dependent variableΨ. Equation (4) is the second order linear differential equation in terms of the new variables Ψ and *x*, and is the master equation for the system (2). Also the equation (4) to be solved to find Ψ, i.e the metric function μ.

3. RESULTSAND DISCUSSION

It is possible to express the solution of (4) in terms of special functions namely theGegenbauer functions. However that form of thesolution is not particularly useful because of the analytic complexity of the specialfunctions involved. In addition the role of parameters of physical interest, such asthe spheroidal parameter *K*, is lost or obscured in the representation as Gegenbauerfunctions. The representation of the solutions in a simple form is necessary for adetailed physical analysis. Consequently we attempt to obtain a general solution to the differential equation (4) in a series form using the method of Frobenius. Later we will indicate that it is possible to extract solutions in terms of polynomials andalgebraic functions for particular parameter values as demonstrated by Komathiraj and Maharaj (2010).

As the point $x = 0$ is the regular point of the differential equation (4), there are two linearly independent solutions. Thus the general solution to (4) can be assumed by the method of Frobenius as

$$\Psi(x) = \sum_{\mathrm{i}=0}^{\infty} a_i x^i \qquad (5)$$

wherea_i are constants. For a legitimate solution we need to determine the coefficients. On substituting the series (5) in (4), we obtain after simplification

$$K(K-\alpha-1)a_0 + Kx(K-\alpha-2)a_1 + 2(1-K)a_2 + 6x(1-K)a_3 + \sum_{i=2}^{\infty}\{(1-K)(i+1)(i+2)a_{i+2} + K[K-\alpha-1+i(i-2)]a_i\}x^i = 0$$

inincreasing powers of *x*. For this equation to be valid for all *x* in the interval ofconvergence we require

$$\sum_{i=2}^{\infty}\{(1-K)(i+1)(i+2)a_{i+2} + K[K-\alpha-1+i(i-2)]a_i = 0$$

which is the linear recurrence relation governing the structure of the solution. The recurrence relation consists of variable, rational coefficients. It does not fall in the known class of difference equations and has to be solved from first principles. It is possible to solve using the principle of mathematical induction. All the even coefficients can be written in terms of the coefficient a_0. These coefficients generate a pattern

$$a_{2i} = (K|K-1)^i \frac{1}{(2i)!}\prod_{q=1}^{i}[K-\alpha-1+(2q-2)(2q-4)]\, a_0$$

We can obtain a similar formula for the odd coefficients as

$$a_{2i+1} = (K|K-1)^i \frac{1}{(2i+1)!}\prod_{q=1}^{i}[K-\alpha-1+(2q-1)(2q-3)]\, a_1$$

Hence the difference equation has been solved and all nonzero coefficients are expressible in terms of the leading coefficients $a_0$and a_1. From these two patterns and (5) we establish that

$$\Psi(x) = a_0\left(1 + \sum_{\mathrm{i}=1}^{\infty}(K|K-1)^i \frac{1}{(2i)!}\prod_{q=1}^{i}[K-\alpha-1+(2q-2)(2q-4)]\,x^{2i}\right)$$
$$+ a_1\left(1 + \sum_{\mathrm{i}=1}^{\infty}(K|K-1)^i \frac{1}{(2i+1)!}\prod_{q=1}^{i}[K-\alpha-1+(2q-1)(2q-3)]\,x^{2i+1}\right)$$

Thus we have found the general series solution to the differential equation (4) for the choice of the one of the metric function and the anisotropic factor. The solution is expressed in terms of a series with real arguments unlike the complex arguments given by software packages. The series converge if there exists a nonnegative value for the radius of convergence. Note that the radius of convergence of the series is not less than the distance from the centre (x = 0) to the nearest root of the leading coefficient of the differential equation (4). Clearly this is possible for a wide range of values for K.It is interesting to observe that the series in terminates for restricted values of theparameters α and K. This will happen when α and K takes on specific integer values.Utilising this feature it is possible to generate solutions in terms of elementary functionsby determining the specific restriction on α and K for a terminating series. Solutionsin terms of polynomials and algebraic functions can be found. We use the recurrencerelation, rather than the series to find the elementary solutions as this issimpler.

Two classes of solutions in terms of elementary functions for (4) are possible from the above series form. The first category of solutions for $\Psi(x)$ is given by

$$\Psi_1(x) = A\sum_{j=0}^{n}(-\beta)^j \frac{(n+j-2)!}{(n-j)!\,(2j)!}x^{2j} + B(1-K+Kx^2)^{\frac{3}{2}}\sum_{j=0}^{n-2}(-\beta)^j \frac{(n+j)!}{(n+j-2)!\,(2j+1)!}x^{2j+1}$$

for the values

$$\beta = 4 - \frac{4}{4n(n-1)-\alpha}$$

$$K-\alpha = [2-(2n-1)^2]$$

The second category of solution for $\Psi(x)$ has the form

$$\Psi_2(x) = A\sum_{j=0}^{n}(-\gamma)^j \frac{(n+j-1)!}{(n-j)!\,(2j+1)!}x^{2j+1} + B(1-K+Kx^2)^{\frac{3}{2}}\sum_{j=0}^{n-1}(-\gamma)^j \frac{(n+j)!}{(n-j-1)!\,(2j)!}x^{2j}$$

with the values

$$\gamma = 4 - \frac{4}{4n^2 - 1 - \alpha}$$

$$K - \alpha = 2(1-2n)^2$$

In the above two categoriesof solutions *A* and *B* are arbitrary constants.

DISCUSSION

One of the original reasons for studying anisotropic matter was to generate models that permit redshifts higher than the critical redshift of isotropic matter. Observational results indicate that certain isolated objects haveredshifts higher than critical redshiftof isotropic matter.We have found new solutions to the Einstein field equations for an anisotropic fluid spherebyutilizing the method of Frobenius for an infinite series; a particular form for one of thegravitational potentials was assumed and the anisotropic factor was specified. Thesesolutions are given in terms of special functions. For particular values of the parametersinvolved it is possible to write the solution in terms of elementary functions: polynomialsand products of polynomials and algebraic functions. The anisotropic factor may vanishin the solutionsisotropic solutionscan be regained. Thus this approachhas the advantage of necessarily containing isotropic stellar solution.It is also important to observe that the magnitude of the anisotropic factor is a nonzero function in general for many solutions found previously. Hence this class of solutions is generally anisotropic and does not have an isotropic limit. An analogous situation risesin Einstein-Maxwell solutions modelling charged relativistic stars in whichthe electric field is always present. An example of such a charged star isgiven byHansraj and Maharaj (2006). The simple form of the solutions found facilitates the analysis of the physical features of an anisotropic fluid sphere. The gravitational potentials are finite at the centre *r = 0* and at the boundary *r* = *R*. These functions are continuous and well behaved in the interior of the relativistic star. The radial pressure is continuous and well behaved in the interior of the star. Also the radial pressure is greater than zero in the interval (0, *R*), regular at the centre, and vanishes at the boundary. In general the tangential pressure is not zero at the boundary of the star which is different from the radial pressure.

4. CONCLUSION

The main objective of this work was to find exact solutions to the anisotropic fluid sphere which can be used to describe a relativistic dense star. Solutions of the complicated system of nonlinear partial differential equations were sought by specifying physically reasonable forms for one of the gravitational potentials and the anisotropic factor. A number simple solutions to the system, which we believe to be physically reasonable, were obtained explicitly in terms of elementary functions. It was also possible to find other categories of solutions by specifying other types of spatial geometries.

5. REFERENCES

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